1 1D waves

1.1 Wave propagation

$\frac{\partial u^2}{\partial t^2} = c^2 \frac{\partial u^2}{\partial x^2}$

$u(x,t) = f(x+ct) + g(x-ct)$

$\epsilon = \frac{\partial u}{\partial x}$

$v = \frac{\partial u}{\partial t}$

1.2 Impedance

$R = \frac{\rho_2 c_2}{\rho_1 c_1}$

$f_T = \frac{2R}{R+1} f_I$

$f_R = \frac{R-1}{R+1} f_I$

$f_I + f_R = f_T$

1.3 Energy

$E_{pot} = \frac{S}{2}\int_0^L \sigma\epsilon dx$

$E_{kin} = \frac{S}{2}\int_0^L \rho v^2 dx$

1.4 Harmonic waves

$u(x,t) = A\cos(kx \pm \omega t) = Ae^{i(\omega t - kx)}$

$\omega = \frac{2\pi}{T} = 2\pi f$

$\lambda = cT$

$k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$

$c = \frac{\omega}{k} = \frac{\lambda}{T}$

2 3D waves

2.1 Wave propagation

$\left[(\lambda+\mu)\underline{p}\otimes\underline{p} + \mu(\underline{p}.\underline{p})\underline{I}\right]\underline{d} = \rho c^2 \underline{d}$

$\underline{d} = d\underline{p} \rightarrow C_L = \sqrt{\frac{\lambda+2\mu}{\rho}} = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}}$

$\underline{d}.\underline{p} = 0 \rightarrow C_T = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{E}{2\rho(1+\nu)}}$

2.2 Rayleigh waves

$u_1 = Ae^{-bx_2}e^{ik(x_1-ct)}$

$u_2 = Be^{-bx_2}e^{ik(x_1-ct)}$

$u_3 = 0$

2.3 Helmoltz decomposition

$\underline{u} = \underline{u}_L + \underline{u}_C = \underline{\nabla}\varphi + \underline{rot}\underline{\psi}$

3 Linear Elastic Fracture Mechanics

3.1 Griffith's energy criterion (Global)

$\frac{dE}{dA} = \frac{dE_{pot}}{dA} + \frac{dW}{dA}$

$\sigma_f = \sqrt{\frac{2E\gamma_s}{\pi a}}$

$a_{Griffith} = \frac{2E\gamma_s}{\pi\sigma^2}$

3.2 Energy release rate

$G = -\frac{dE_{pot}}{dA} = \frac{1}{B}\frac{dU}{da} = \frac{1}{B}\frac{d}{da}(W - U_{el})$

$C = \frac{\Delta}{P}$

$G = \frac{P^2}{2B}\frac{dC}{da}$

3.3 Airy function

$\nabla^4\phi = 0$

$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}$

$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}$

$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$

3.4 Irwin's solution (Local)

3.4.1 Mode I

$\sigma_{ij}(r,\theta) \simeq \frac{K_I}{\sqrt{2\pi r}} f_{ij}(\theta)$

$f_{xx} = \cos\left(\frac{\theta}{2}\right)\left[1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$

$f_{yy} = \cos\left(\frac{\theta}{2}\right)\left[1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$

$f_{xy} = \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)$

$f_{zz}(\text{Plane strain}) = \nu(f_{xx} + f_{yy})$

$f_{zz}(\text{Plane stress}) = f_{xz} = f_{yz} = 0$

3.4.2 Mode II

$\sigma_{ij}(r,\theta) \simeq \frac{K_{II}}{\sqrt{2\pi r}} f_{ij}(\theta)$

$f_{xx} = -\sin\left(\frac{\theta}{2}\right)\left[2 + \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)\right]$

$f_{yy} = \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)$

$f_{xy} = \cos\left(\frac{\theta}{2}\right)\left[1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$

$f_{zz}(\text{Plane strain}) = \nu(f_{xx} + f_{yy})$

$f_{zz}(\text{Plane stress}) = f_{xz} = f_{yz} = 0$

3.5 K values

Crack, length $2a$, in an infinite plate:

$K_I = \sigma\sqrt{\pi a}$

Same shape, with a symmetric stress distribution $\sigma(x)$ along y:

$K = 2\sqrt{\frac{a}{\pi}}\int_0^a \frac{\sigma(x)}{\sqrt{a^2-x^2}} dx$

Crack, length a, on the edge of a semi-infinite plate:

$K_I = 1.12\sigma\sqrt{\pi a}$

Crack of length $2a$ in an infinite plate, with an angle β:

$K_I = \sigma\cos^2\beta\sqrt{\pi a}$

$K_{II} = \sigma\sin\beta\cos\beta\sqrt{\pi a}$

Central penny-shaped crack, radius a, in an infinite body:

$K_I = \frac{2}{\pi}\sigma\sqrt{\pi a} = 2\sigma\sqrt{\frac{a}{\pi}}$

Same shape, with a symmetric stress distribution $\sigma(r)$ along y:

$K = \frac{2}{\sqrt{\pi a}}\int_0^a \frac{r\sigma(r)}{\sqrt{a^2-r^2}} dr$

Decomposition of a problem into subparts: $K_I^{tot} = K_I^A + K_I^B + K_I^C \ldots$

3.6 K-G relation

$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$

Plane strain : $E' = \frac{E}{1-\nu^2}$

Plane stress : $E' = E$

3.7 Mixed-Mode Fracture

Principle of Local Symmetry:

$K_{II}^*(\alpha^*) = 0$

Maximum Hoop Stress:

$\sigma_{\theta\theta}(\alpha) \leq \sigma_{\theta\theta}(\alpha^*), \forall \alpha$

Maximum Energy Release Rate Criterion:

$G(\alpha) \leq G(\alpha^*), \forall \alpha$

4 Brittle materials

$S(V,\sigma) = \Pi_{i=1}^{N} S(\Delta V_i, \sigma)$

2-parameters Weibull distribution:

$F(V,\sigma) = 1 - S(V,\sigma) = 1 - \exp\left[-\int_V \left(\frac{\sigma(V)}{\sigma_0}\right)^m \frac{dV}{V_0}\right] = 1 - \exp\left[-k_m \frac{V}{V_0}\left(\frac{\sigma_{max}}{\sigma_0}\right)^m\right]$

Tensile: $k_m = 1$

3-point bending: $k_m = \frac{1}{2(m+1)^2}$

4-point bending: $k_m = \frac{mL_i + L_o}{2L_o(m+1)^2}$

5 Plasticity in Fracture Mechanics

$\sigma_{VM} = \frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right]^{0.5}$

$K_{IC} = \sqrt{EG} = \sqrt{E(2\gamma_s + G_{pl})}$

1st order approximation: $\sigma_{yy} = \sigma_{ys}$

$r_p = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_{ys}}\right)^2$

2nd order approximation:

$r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_{ys}}\right)^2$

J-integral (for pulling in the y-direction):

$J = \int_\Gamma (w\,dy - T_i \frac{\partial u_i}{\partial x} ds)$

6 Fatigue crack growth

Paris law:

$\frac{da}{dN} = C\Delta K^m$ with m between 2 and 4.